

Bell inequalities as constraints on unmeasurable correlations

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Abstract

The interpretation of the violation of Bell-Clauser-Horne inequalities is revisited, in relation with the notion of *extension* of QM predictions to *unmeasurable correlations*. Such extensions are compatible with QM predictions in many cases, in particular for observables with compatibility relations described by tree graphs. This implies classical representability of any set of correlations $\langle A_i \rangle$, $\langle B \rangle$, $\langle A_i B \rangle$, and the equivalence of the Bell-Clauser-Horne inequalities to a non void intersection between the ranges of values for the *unmeasurable correlation* $\langle A_1 A_2 \rangle$ associated to different choices for B . The same analysis applies to the Hardy model and to the “perfect correlations” discussed by Greenberger, Horne, Shimony and Zeilinger. In all the cases, the dependence of an unmeasurable correlation on a set of variables allowing for a classical representation is the only basis for arguments about violations of locality and causality.

Introduction and results

The implications of the violation of Bell inequalities [1] in Quantum Mechanics (QM) are still controversial. On one side, their logical and probabilistic nature has been recognized and discussed [2], [3], [4] and their fundamental meaning traced back [5] to Boole's general notion of *conditions of possible experience* [6]. In this perspective, their violation, which excludes representability of quantum mechanical predictions by a classical probability theory, shows general and fundamental differences between the classical and the quantum notion of event.

On the other, the violation of Bell inequalities is interpreted by many authors in terms of “non local properties” of QM; an extensive review of such positions is contained in Ref. [7] and a discussion of their basic logical steps is in Refs. [8], [9], [10], [11].

In our opinion, in general, the use of Bell inequalities for the analysis of QM and of its interpretation is in a sense incomplete because their violations simply amounts to a negative result, i.e. the inconsistency between QM and *the whole set* of assumptions entering in their derivation.

The present paper is an attempt to reconsider the situation from a more constructive point of view, beginning with QM predictions, trying to represent them by classical probability models in a sequence of steps and asking which step may fail.

We recall that adopting a classical probability model exactly amounts to assuming that *all variables together take definite, even if possibly unmeasurable, values*, with definite probabilities.

Since a classical probability model is equivalent to its set of predictions for *all* correlations, while QM only predicts correlations between *compatible* observables, classical representability also amounts to an extension of QM predictions to *unmeasurable correlations*, e.g., correlations between two components of the spin *of the same particle*, or polarization *of the same photon* along different directions.

A central role is therefore played by the notion of *extension of QM predictions to unmeasurable correlations*. We shall use basic and general constraints on such *extensions* to discuss the violation of Bell inequalities in terms of actual properties of existing (partial) extensions rather than in terms of incompatibility between assumed principles. In particular, this will allow for a definite answer to the question *what precisely is “influenced”* in arguments on non-local effects in QM.

The extension problem outlined above has been studied in general in [12]. One of the result (see Sect.1) is that classical representability *always holds* for yes/no observables with compatibility relations described by *a tree graph*, i.e. a graph in which any two points are connected by exactly one path, with points representing observables and links predicted correlations.

In the case of four yes/no observables, A_1, A_2, B_1, B_2 , A_i compatible with B_j , this implies that, for fixed B , any pair of probabilities for the subsystems A_i, B $i = 1, 2$, assigning

the same probability to the outcomes of B , admits extensions to a probability on the whole system A_1, A_2, B , each giving a definite value to the possibly unmeasurable correlation $\langle A_1 A_2 \rangle$. As we will see, the results on tree graphs also imply that *if a value can be assigned to $\langle A_1 A_2 \rangle$ consistent with two choices B_1, B_2 for B , then the whole system A_i, B_j admits a classical representation.*

Since the converse is obvious, it follows that the existence of a classical representation for given set of expectations $\langle A_i \rangle$, $\langle B_i \rangle$ and correlations $\langle A_i B_j \rangle$ is equivalent to the possibility of giving a value to $\langle A_1 A_2 \rangle$ which is *consistent with the two choices for B .*

The analysis of QM models with four yes/no observables is therefore reduced in general to the consistency (i.e. a non void intersection) between the ranges of the (hypotetical, unmeasurable) values for the correlation $\langle A_1 A_2 \rangle$ allowed by (measurable) given correlations of A_1 and A_2 with B_1 and B_2 .

With respect to the same conclusion obtained by Fine [2], we stress that probabilistic descriptions of three-observable subsystem *automatically exist* and only their compatibility is in question; moreover, only a repeated application of the results for tree graphs is required by our argument.

In Sect.2, the possible range for $\langle A_1 A_2 \rangle$, depending on the measurable correlations $\langle A_i B \rangle$, is characterized in terms of elementary three-observable inequalities. QM states which violate Bell inequalities are shown to give rise to disjoint intervals for the admissible values of $\langle A_1 A_2 \rangle$ in automatically existing probabilistic models for A_1, A_2, B_1 and A_1, A_2, B_2 . This clearly shows that classical representability of such states exactly fails in the attribution of a value to an unmeasurable correlation.

Actually, since BCH inequalities, more precisely, the eight inequalities discussed by Fine [2], are equivalent to classical representability, their violation exactly amounts to the inconsistency of such an attribution.

The same discussion and result apply to the states introduced, on the same set of observables, by Hardy [8] and exploited in Stapp's work [9], discussed by Mermin [10]. In particular, Stapp's and Mermin's discussion exactly concerns partial extensions and the origin of the inconsistency of the ranges of value for an unmeasurable correlation $\langle A_1 A_2 \rangle$ associated as above to the correlations of A_i with B_1 and B_2 respectively.

In Sect.3, the experiment proposed by Greenberger, Horne, Shimony and Zeilinger (GHSZ) [13] is analyzed along the same lines. The QM correlations defined by the GHSZ state, between observables $A_i, B_i, C_i, i = 1, 2$, are shown to extend to *all correlations* within the sets of observables A_i, B_i, C_1 and A_i, B_i, C_2 ; both extensions give *unique values* for an *unmeasurable correlation*, $\langle A_1 A_2 B_1 B_2 \rangle = \pm 1$, and the sign depends on the choice between C_1 and C_2 . Again, only the attribution of a value to an unmeasurable correlation depends on the choice of an additional observable.

In the last Section we will comment on the fact that such a dependence is the only basis for arguments about non local and non causal effects.

1 Extensions of QM predictions

A basic notion is that of yes/no observables, denoting physical devices producing, in each measurement, i.e. in each application to a physical system, two possible outcomes; by *experimental setting* we will denote a collection of yes/no observables.

Within an experimental setting, an (*experimental*) *context* will denote a (finite) set of observables which “can be measured together on the same physical system”, in the precise sense that their joint application to a physical system gives rise to a statistics of their joint outcomes given by a classical probability, i.e., by a normalized measure on the Boolean algebra freely generated by them. Observables appearing together in some context will be called *compatible*; their correlations will be called *measurable*, or *observable*.

In the following, Quantum Mechanics will be interpreted as a theory predicting probabilities associated to experimental contexts defined by sets of commuting projections. Joint statistics of non compatible observables are not defined in the above setting; correlations between incompatible observables will be called *unmeasurable*, or *unobservable*.

In particular, we will consider the *Bell-Clauser-Horne (BCH) experimental setting*, consisting of two pairs (A_1, A_2) , (B_1, B_2) , of incompatible *yes/no* observables, taking values in $\{0, 1\}$; the A_i are compatible with the B_j , i.e. they can be measured together, in the sense introduced above. Such variables may be interpreted in terms of spin components or photon polarizations, each pair referring to one of a pair of particles, possibly in space-like separated regions.

The experimental contexts given by QM consist, in the BCH setting, of the four pairs $\{A_i, B_j\}$; probabilities are associated to contexts, see Proposition 1.3 below, by the observed relative frequencies $\langle A_i \rangle$, $\langle B_j \rangle$, and $\langle A_i B_j \rangle$. QM predictions for the BCH setting precisely consist in probabilities on contexts, given by the spectral representation for the corresponding sets of commuting projectors.

Such a *partial* nature of QM predictions is a very general fact since the (ordinary) interpretation of QM precisely consists of a set of probability measures, each defined, by the spectral theorem, on the spectrum of a commutative subalgebra of operators in a Hilbert space. Such a structure has been formalized in [12] as a *partial probability theory* on a *partial Boolean algebra*.

A basic problem for the interpretation of QM is *the necessity* of such partial structures; in other words, *whether they admit a classical representation*, i.e. common probabilistic classical description in a probability space (X, Σ, μ) , where X is a set, Σ a σ -algebra of measurable (with respect to μ) subsets and μ is a probability measure on X and all (yes/no) variables are described by characteristic functions of measurable subsets. Clearly, such a notion covers all kinds of “hidden variables” theories, all ending in the attribution of values to all the observables, with definite probabilities.

Extensions have been discussed in general in [12]. Non trivial extensions, describing QM predictions through a reduced number of contexts, have been shown to arise in some

generality and some of the results can be described in terms of graphs:

Proposition 1.1. *Consider any set of probabilities p_i on a set of yes/no observables A_i and correlations p_{ij} on a subset of pairs A_i, A_j , defining a probability on each pair A_i, A_j , with $p_i = \langle A_i \rangle$, $p_j = \langle A_j \rangle$, $p_{ij} = \langle A_i A_j \rangle$; describing observables A_i as points and the above pairs as links in a graph, any subset of predictions associated to a tree subgraph admits a classical representation.*

Proposition 1.2. *the same holds with yes/no observables A_i substituted by free Boolean algebras \mathcal{A}_i , p_i by probabilities on \mathcal{A}_i , p_{ij} by probabilities on the Boolean algebra freely generated by the union of the sets of generators of \mathcal{A}_i and \mathcal{A}_j .*

Propositions 1.1 and 1.2 are proven in [12] by induction on the number of links of the correlation tree, by an explicit construction of a probability on the Boolean algebra generated by two algebras in terms of conditional probabilities with respect to their intersection. The resulting probabilities are in general not unique.

The above Propositions can be interpreted as providing automatic conditions for the realization of the possibility advocated by Einstein, Podolsky and Rosen [14] to attribute a value to correlations between incompatible observables consistently with the QM predictions for the remaining, measurable, correlations; actually, they do not make reference to QM and give conditions on compatibility relations allowing the extension of any set of predictions.

It is also useful to recall the following elementary fact [12], which implies that it is sufficient to analyze correlations since they completely define a probability measure.

Proposition 1.3. *In a classical representation for n observables A_1, \dots, A_n , the probability measure is completely defined by all expectation values $\langle A_i \rangle$ together with all possible correlations $\langle A_i A_j \rangle$, $\langle A_i A_j A_k \rangle, \dots, \langle A_1 \dots A_n \rangle$.*

In the BCH setting, the two subsets $\{A_1, A_2, B_1\}$, $\{A_1, A_2, B_2\}$, with the correlations $\langle A_j B_i \rangle$ $j = 1, 2$, are described by tree graphs (see Fig.1 (a),(b)) and admit therefore classical representations. It follows that all QM predictions can be reproduced by a pair of classical probability theories, one for the Boolean algebra freely generated by A_1, A_2, B_1 the other for A_1, A_2, B_2 ; both sets of variables include A_1 and A_2 and give rise to predictions for the unmeasurable correlation $\langle A_1 A_2 \rangle$.

In other terms, *given a choice for B* , one may consistently speak of the correlation $\langle A_1 A_2 \rangle$. Clearly, the existence of a classical representation for all the predictions in the BCH setting implies the existence of a value for $\langle A_1 A_2 \rangle$ compatible with the different choices of B .

On the other hand, if it is possible to give a value to $\langle A_1 A_2 \rangle$ independently of the choice B_1, B_2 for B , this defines, by Proposition 1.3, a probability on the Boolean algebra \mathcal{A}_{12}

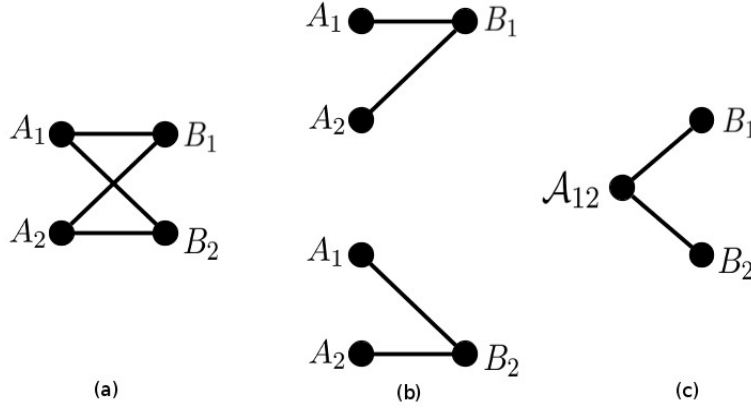


Figure 1: (a) BCH compatibility graph; (b) two tree subgraphs; (c) tree graph corresponding to extension of the measure on the algebra generated by A_1, A_2

generated by A_1, A_2 ; then, the application of Proposition 1.2 to $\mathcal{A}_{12}, \{B_1\}, \{B_2\}$ implies classical representability of all the predictions for the BCH setting (see Fig.1 (c)).

Classical representability and BCH inequalities (in the sense of Ref.[2]) are therefore equivalent to the possibility of attributing a common value to the unmeasurable correlation $\langle A_1 A_2 \rangle$ in classical models for A_1, A_2, B_1 and A_1, A_2, B_2 .

In the next Section, we shall compute explicitly the range of $\langle A_1 A_2 \rangle$ allowed by the QM predictions for a pair of spin variables, in the zero total angular momentum state and for the state introduced by Hardy [8].

2 Explicit Constraints for systems of three and four observables

In this section we first recall simple constraints, for three $\{0, 1\}$ valued random variables A_1, A_2, B , satisfied by $\langle A_1 A_2 \rangle$ given $\langle A_i \rangle$, $\langle B \rangle$ and $\langle A_i B \rangle$. We then show that, for the predictions given by both the QM states discussed by Bell and Hardy [8], such constraints give rise, for different choices of B , to incompatible values for $\langle A_1 A_2 \rangle$.

The possible ranges of correlations in a classical model for $\{A_1, A_2, B\}$ are given by the Bell-Wigner polytope [4], also discussed in [12]. We only need a subset of the corresponding set of inequalities, namely

$$0 \leq \langle A_1 A_2 \rangle \leq \langle A_1 \rangle \quad (1)$$

$$\langle A_1 B \rangle + \langle A_2 B \rangle - \langle B \rangle \leq \langle A_1 A_2 \rangle \leq \langle A_2 \rangle - \langle A_2 B \rangle + \langle A_1 B \rangle \quad (2)$$

$$\langle A_1 \rangle + \langle A_2 \rangle + \langle B \rangle - \langle A_1 B \rangle - \langle A_2 B \rangle - 1 \leq \langle A_1 A_2 \rangle \quad (3)$$

Eq. (1) is trivial; the first relation in eq. (2) immediately follows from $\langle (A_1 - B)(A_2 - B) \rangle \geq 0$, which holds since for $B = 1$ both factors are non positive, for $B = 0$ they are both non negative; moreover, the first relation in eq. (2) implies the second by interchanging B and A_2 and eq. (3) by interchanging B and $1 - B$. The complete set of constraints [4] includes three additional inequalities; they are obtained from the above relations through the interchange of A_1 and A_2 and the substitution of B with 1 in eqs. (1), (2) (and are not relevant for our analysis).

The constraints given by eqs. (1), (2) (3), applied to QM predictions for mean values and *observable* correlations, may give rise to ranges of values for the unobservable correlation $\langle A_1 A_2 \rangle$ which are disjoint for different choices of B .

Consider in fact a system of two spin 1/2 particles in the singlet state $\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$ and associate A_1 and A_2 with the projectors on the spin up state along directions \hat{x} and \hat{z} for the first particle and B_1 with the projector on the spin down state along the direction $\frac{\hat{x} + \hat{z}}{\sqrt{2}}$ for the second particle, namely

$$\frac{1 + \sigma_x}{2}, \quad \frac{1 + \sigma_z}{2}, \quad \frac{1}{2} - \frac{\tau_x + \tau_z}{2\sqrt{2}}, \quad (4)$$

where σ_i 's and τ_i 's are Pauli matrices acting, respectively, on the first and on the second subsystem. Quantum mechanical predictions give

$$\langle A_1 \rangle = \langle A_2 \rangle = \langle B_1 \rangle = \frac{1}{2}, \quad (5)$$

$$\langle A_1 B_1 \rangle = \langle A_2 B_1 \rangle = \frac{1}{4} + \frac{\sqrt{2}}{8}, \quad (6)$$

Denoting by $\langle \rangle_1$ correlations in a probabilistic model for $\{A_1, A_2, B_1\}$ eqs. (1) and (2) give

$$\frac{1}{4} < \frac{\sqrt{2}}{4} \leq \langle A_1 A_2 \rangle_1 \leq \frac{1}{2}. \quad (7)$$

If B_1 is substituted by B_2 , representing the projector

$$\frac{1}{2} - \frac{\tau_x - \tau_z}{2\sqrt{2}}, \quad (8)$$

QM predictions give $\langle B_2 \rangle = \frac{1}{2}$ and

$$\langle A_1 B_2 \rangle = \frac{1}{4} + \frac{\sqrt{2}}{8}, \quad \langle A_2 B_2 \rangle = \frac{1}{4} - \frac{\sqrt{2}}{8}. \quad (9)$$

Eqs. (1) and (2) then give, for the unmeasurable correlation $\langle \rangle_2$ in probabilistic models for $\{A_1, A_2, B_2\}$

$$0 \leq \langle A_1 A_2 \rangle_2 \leq \frac{1}{2} - \frac{\sqrt{2}}{4} < \frac{1}{4} . \quad (10)$$

It follows that the value of $\langle A_1 A_2 \rangle$ must belong to one of two *disjoint intervals*, depending on whether one is considering the probability space for $\{A_1, A_2, B_1\}$ or that for $\{A_1, A_2, B_2\}$, i.e. whether one chooses to measure B_1 or B_2 on the second subsystem. Therefore, the validity of QM predictions for compatible observables *does not allow* to assign a definite value to the correlation $\langle A_1 A_2 \rangle$, independent of the choice of the measurement to be performed on the second subsystem.

The second example is Hardy's experiment [8], also exploited in Stapp's [9] and Mermin's [10] discussion; we shall refer to Mermin's notation. As in the previous example, we have two spin 1/2 particles and four yes/no observables, denoted by $L1, L2$ and $R1, R2$, acting respectively on a "left" and a "right" particle. The state is described, up to a normalization factor, by the vector

$$|\Psi\rangle = |L1+, R1-\rangle - |L2-, R2+\rangle \langle L2-, R2+| L1+, R1-\rangle , \quad (11)$$

where, e.g, $|L1+, R1-\rangle$ indicates a simultaneous eigenstate of the commuting observables $L1$ and $R1$ with eigenvalue 1 on the left and -1 on the right.

The variables A_i , $i = 1, 2$, are associated with the propositions "the result of the measurement of Ri is $+1$ ", the same for B_i and Li ; as before, A_i and B_i take values in $\{0, 1\}$. Independently of the specific expressions for Li and Ri , QM predicts the following correlations (corresponding to eqs.(6)-(9) in Ref. [10]):

$$1 - \langle A_2 \rangle - \langle B_1 \rangle + \langle A_2 B_1 \rangle = 0 , \quad (12)$$

$$\langle A_2 \rangle - \langle A_2 B_2 \rangle = 0 , \quad (13)$$

$$\langle A_1 B_2 \rangle = 0 , \quad (14)$$

$$\langle A_1 \rangle - \langle A_1 B_1 \rangle > 0 , \quad (15)$$

In eq. (15), > 0 is equivalent to $\neq 0$ in Ref. [10] since $\langle A_1 \rangle \geq \langle A_1 B_1 \rangle$ always holds.

Eqs. (3), (12) and (15) give, in all classical models for $\{A_1, A_2, B_1\}$

$$\langle A_1 A_2 \rangle_1 \geq \langle A_1 \rangle + \langle A_2 \rangle + \langle B_1 \rangle - \langle A_1 B_1 \rangle - \langle A_2 B_1 \rangle - 1 = \langle A_1 \rangle - \langle A_1 B_1 \rangle > 0 ; \quad (16)$$

on the other hand, in classical models for $\{A_1, A_2, B_2\}$, eqs. (2), (12) and (15) give,

$$\langle A_1 A_2 \rangle_2 \leq \langle A_2 \rangle - \langle A_2 B_2 \rangle + \langle A_1 B_2 \rangle = 0 , \quad (17)$$

which also implies, by eq. (1),

$$\langle A_1 A_2 \rangle_2 = 0 . \quad (18)$$

As in the previous example, no value for the unmeasurable correlation $\langle A_1 A_2 \rangle$ is compatible with different choices of B_i . A peculiarity of this case is that in classical models for $\{A_1, A_2, B_2\}$ the correlation $\langle A_1 A_2 \rangle$ is completely fixed (to 0) by the constraints imposed by measurable correlations. Attributing the 0 value is actually equivalent to asserting the logical implication $A_1 \rightarrow A_2^c$, i.e. “*if the measurement of $R1$ gives +1, then the measurement of $R2$ gives -1*”; such an implication *holds in all classical models for $\{A_1, A_2, B_2\}$ and fails in all classical models for $\{A_1, A_2, B_1\}$.*

3 The perfect correlation model of GHSZ

In this Section, the above analysis is applied to the case considered in [13] and discussed in [15] within a general framework unifying Bell and Kochen-Specker kinds of results.

With a slight modification of the above terminology, the experimental setting involved in the GHSZ model consists of six yes/no observables, which can be described as A_i, B_i, C_i , $i = 1, 2$, taking values 1 and -1 ; equivalently, by the propositions P_i, Q_i, R_i respectively asserting $A_i = 1, B_i = 1, C_i = 1$. The experimental contexts, i.e. the sets of compatible observables are given by the possible choices of (at most) one A , one B and one C .

In Refs. [13] and [15] observables associated to different letters are interpreted as measured in distant, space-like separated regions. The above observables reproduce the example of Ref. [13], Sect. III, with the identification $A_i = A(\phi_i)B(0)$, $B_i = C(\phi_i)$, $C_i = D(\phi_i)$, with $\phi_1 = 0$ $\phi_2 = \pi/2$; in Mermin’s spin notation, they should be read as $A_1 = -\sigma_x^1$, $A_2 = -\sigma_y^1$, $B_1 = \sigma_y^2$, $B_2 = \sigma_x^2$, $C_1 = \sigma_y^3$, $C_2 = \sigma_x^3$.

In general, quantum mechanical predictions consist in probability assignments within each of the eight contexts defined by the choice of three indexes i ; e.g. a context is given by the choice A_1, B_1, C_1 , and the associated predictions by a probability on the Boolean algebra freely generated by the propositions P_1, Q_1, R_1 .

The crucial point of GHSZ and Mermin’s analysis is the observation that, for suitable states, QM predictions give “perfect correlations” (each correlation involving observables in a fixed context), which are not compatible with any (context-independent) assignment of values to the variables A_i, B_i, C_i .

In fact, the state considered by GHSZ and Mermin can be written, for the Mermin spin variables σ_k^i , in the usual notation referring to the eigenvalues of σ_3^i ,

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|+++\rangle - |--\rangle) \quad (19)$$

and gives the correlations

$$\langle A_1 B_1 C_1 \rangle = -1, \quad \langle A_2 B_2 C_1 \rangle = -1, \quad (20)$$

$$\langle A_2 B_1 C_2 \rangle = -1, \quad \langle A_1 B_2 C_2 \rangle = +1. \quad (21)$$

Values $a_i, b_i, c_i = \pm 1$ assigned to the above variables and reproducing the above correlations would satisfy the relations

$$a_1 b_1 c_1 = -1, \quad a_2 b_2 c_1 = -1, \quad a_2 b_1 c_2 = -1, \quad a_1 b_2 c_2 = +1, \quad (22)$$

which have no solution since their product gives 1 for the l.h.s., -1 for the r.h.s..

Following the program outlined above and using the notions and results of Sect. 1, we will extend the GHSZ and Mermin argument to show:

- i)* the correlations eqs. (20), (21) *and all the other QM predictions* given by the GHSZ state, eq. (19), extend to classical probabilities p_1, p_2 on the *two* subsystems defined by $\{A_1, A_2, B_1, B_2, C_1\}$ and $\{A_1, A_2, B_1, B_2, C_2\}$
- ii)* unique and different values, ± 1 , are given by such p_1 and p_2 to the correlation $\langle A_1 A_2 B_1 B_2 \rangle$, all the other correlations between the A and B observables admitting common values;
- iii)* pairs of probabilities on the two subsystem introduced in *i)* extend to a probability on the entire system generated by $A_i, B_j, C_k, i, j, k = 1, 2$, exactly if they coincide on the subsystem generated by A_1, A_2, B_1, B_2 .

Again, the conclusion is that *exactly a non observable correlation*, $\langle A_1 A_2 B_1 B_2 \rangle$, *depends* on the choice of the observable C_i .

In order to derive *i) - iii)*, we first observe that *iii)* follows immediately from Proposition 1.2, applied to the Boolean algebra generated by P_i, Q_j and to its correlations with the observables R_1 and R_2 , forming a tree graph.

Concerning *ii)*, observe that eqs. (20) imply, for any probabilistic model reproducing them, $A_1 B_1 C_1 = -1$ and $A_2 B_2 C_1 = -1$ with probability 1, so that, with probability 1, $A_1 B_1 A_2 B_2 = 1$; equivalently, $\langle A_1 B_1 A_2 B_2 \rangle = 1$. In the same way, eqs. (21) imply $\langle A_1 B_1 A_2 B_2 \rangle = -1$; such relations hold therefore for all probabilistic models reproducing, respectively, the QM correlations of the A, B observables with C_1 and C_2 . The existence of probabilities giving common values to all the other correlations follows from the construction below.

The most involved issue is *i)*, which is non trivial since quantum mechanical predictions involve eight different contexts and *i)* states that the predictions given by the state (19) extend to *all correlations* between variables inside each one of *only two* contexts. Notice that Propositions 1.1 and 1.2 do not imply such an extension, since the corresponding set of correlations is not given by a tree graph and in fact not all quantum states admit it, as also implied by the BCH analysis; the exact form of the quantum predictions we are going to extend is therefore important. It is easy to see that the state given by eq. (19) gives rise to the correlations

$$\langle A_i \rangle = \langle B_i \rangle = \langle C_i \rangle = \langle A_i B_j \rangle = \langle A_i C_j \rangle = \langle B_i C_j \rangle = 0, \quad i, j = 1, 2 \quad (23)$$

$$\langle A_1 B_2 C_1 \rangle = \langle A_2 B_1 C_1 \rangle = \langle A_2 B_2 C_2 \rangle = \langle A_1 B_1 C_2 \rangle = 0 . \quad (24)$$

We have to show that the all the correlations given by eqs. (20),(21),(23),(24) involving the observables A_i, B_j, C_1 can be represented by a classical model, and the same for those involving A_i, B_j, C_2 , and that the two models give the same correlations between variables A_i, B_j with the only exception of $\langle A_1 A_2 B_1 B_2 \rangle$.

The first classical model is defined as follows: let A_i, C_1 denote variables taking values 1 and -1 with probability $1/2$, i.e.

$$A_i^2 = 1, \quad C_1^2 = 1, \quad \langle A_i \rangle = \langle C_1 \rangle = 0, \quad \langle A_1 A_2 \rangle = \langle A_i C_1 \rangle = 0 \quad (25)$$

and define

$$B_i \equiv -A_i C_1 . \quad (26)$$

Eqs. (25) and (26) immediately imply eqs. (20) and all eqs. (23),(24) not involving C_2 .

The second model is defined in the same way, by independent variables A_i and C_2 satisfying the same relations as in eqs. (25) and by variables B_i now defined as

$$B_1 \equiv -A_2 C_2, \quad B_2 \equiv A_1 C_2 . \quad (27)$$

Eqs. (21) and all eqs. (23), (24) not involving C_1 immediately follow.

All the correlations defined by the two models between A and B variables follow from eqs. (25), with C_2 replacing C_1 for the second model, (26),(27); in both models $\langle A_1 A_2 \rangle = 0$ by definition,

$$\langle B_1 B_2 \rangle = \pm \langle A_1 A_2 \rangle = 0 \quad (28)$$

and

$$\langle A_i B_1 B_2 \rangle = 0 = \langle B_i A_1 A_2 \rangle ,$$

both expectations reducing to $\pm \langle A_k \rangle$ for some k . The two models give therefore identical predictions for all the correlations between the A and B variables, with the only exception

$$\langle A_1 A_2 B_1 B_2 \rangle = \pm 1 . \quad (29)$$

4 Conclusions

The discussion of Bell inequalities usually begins with their derivation from general *locality* and *reality* principles. Reality principles *include* hypothetical common attributions of values to observables even when they are not compatible according to QM; once such attributions are assumed, any logical or probabilistic consideration automatically concerns extensions of QM predictions to unmeasurable correlations between incompatible observables.

Moreover, arguments about the possibility of considering measurements which *could have been performed* and the assumption that the results of such “unperformed” measurements satisfy the predictions of QM for their correlation with other, compatible and performed, measurements, lead [9] [10] to logical or probabilistic relations between quantum mechanically incompatible observables which precisely amount to partial extensions of QM predictions.

We have shown that, independently of any additional principle or argument, partial extensions of QM predictions are always possible in cases described by tree graphs, that the ranges of values for correlations between quantum mechanically incompatible observables obtained by such extensions depend in general on the considered set of observables and that they may be incompatible for different sets.

As shown by the Hardy and the GHSZ examples, this may happen even when *all extensions*, to a fixed allowed set of observables, of a set of QM predictions give the same *perfect correlations* to quantum mechanically incompatible observables. If the *dependence of such correlations on a set of observables* allowing for a partial extension is not taken into account, the result appears as a contradiction between logical consequences of QM predictions (and of experimental results).

Moreover, precisely unmeasurable correlations depend on a choice in a far or future space-time regions in the discussion of locality and causality principles. In fact, in the BCH setting, identifying B_i with observables measured in a space-like separated region, only and exactly $\langle A_1 A_2 \rangle$ depends on the choice of B_i ; alternatively, if B_i is measured in a future region, such a dependence violates causality in the precise sense that any hypothetical record of the value of $\langle A_1 A_2 \rangle$ depends on a future choice.

All the above “violations” exactly concern extensions of QM predictions to *correlations between quantum mechanically incompatible observables*, e.g., between two different spin components of *the same* particle or polarization directions of *the same* photon.

A discussion of the violation of BCH inequalities in terms of unobservable correlations has also been given in Ref. [17]. A (different, but related) notion of “extension of quantum correlations” is also central in the recent discussion of the Einstein-Podolsky-Rosen notion of reality in Ref. [18].

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